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NOTE

A Bijective Proof of Lassalle's Partition Identity

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We give a bijective proof of the following form of a conjecture in [2, 3]. The special $s=1$ case was proved in the same vein by Simion [2]. For other proofs of this conjecture see [4]. Our notations are the same as in [2, 3]. For an integer partition μ let the length $l(\mu)$ be the number of the parts of μ, $m_i(\mu)$ the number of parts equal to i, $z_\mu = \prod_{i\geqslant 1} i^{m_i(\mu)} m_i(\mu)!$ and $\langle {\mu \atop r} \rangle$ the number of ways to choose r different cells from the diagram of the partition μ taking at least one cell from each row. For any positive integer n let $(a)_n = a(a+1)\cdots(a+n-1)$ and $[n]=\{1, 2, ..., n\}$.

THEOREM 1. *We have*

$$\sum_{|\mu|=n} \frac{n!}{z_\mu} \left\langle {\mu \atop r} \right\rangle X^{l(\mu)-1} \sum_{i=1}^{l(\mu)} \mu_i \binom{\mu_i+s-1}{s-1} = \binom{n}{r} (s+r)_{n-r} \left[(X+s)_r - (X)_r\right]. \tag{1}$$

Recall that the functional graph of an injection $f\colon [n] \to [n+r]$ consists of disjoint cycles and paths. Each path is of form $a_0 \to a_1 \to \cdots \to a_l$, where $f(a_j) = a_{j+1}$ for $0 \leqslant j < l$, with $f^{-1}(a_0)$ empty, and $a_l \in [n+r] \backslash [n]$. Each cycle is of the form $a_0 \to a_1 \to \cdots \to a_l$, where $f(a_j) = a_{j+1}$ for $0 \leqslant j \leqslant l$, with $a_{l+1} = a_0$. Let cyc f be the number of cycles in f. Then the corresponding generating function reads as follows:

$$\sum_{f\colon [n] \to [n+s]} X^{\operatorname{cyc} f} = (X+s)_n \tag{2}$$

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where f is an injection. The above identity seems to be first published by Foata and Strehl [1] and can be readily verified by induction on n as in the well-known case of permutations, which corresponds to $s=0$.

We shall prove that the two sides of (1) are the generating function of pairs (f, R) with respect to the number of cycles of f, where f is an injection from $[n]$ to $[n+s]$ and R is a r-subset of $[n]$ such that each cycle of f has at least one element in R.

Indeed, to construct such a pair (f, R) we can first choose r elements of $[n]$ to form a r-subset R, there are $\binom{n}{r}$ ways to do so; and construct an injection g: $[r] \rightarrow [r+s]$ such that its restriction on $[r]$ is not a permutation of $[r]$, the generating function of such injections is then $(X+s)_r - (X)_r$; finally insert succesively the $n-r$ elements of $[n] \backslash R$ into the above injection g to obtain the injection f: $[n] \rightarrow [n+s]$, there are $(s+r)_{n-r}$ ways to insert the $n-r$ elements into any above injection g. This gives the right-hand side of (1).

On the other hand, we can construct such a pair (f, R) as follows: construct first a permutation σ on $[n]$, for any partition μ of n there are $n!/z_\mu$ such permutations of type μ; choose r elements of $[n]$ to form the subset R such that each cycle of σ has at least one element in R, there are $\left\langle {\mu \atop r} \right\rangle$ ways to do so; choose any cycle $(a, \sigma(a), ..., \sigma^{\mu_i - 1}(a))$ of σ with length μ_i to form s paths:

$$
\begin{aligned}
a \rightarrow \sigma(a) \rightarrow \cdots \sigma^{l_1 - 1}(a) &\rightarrow n+1, \\
\sigma^{l_1}(a) \rightarrow \sigma^{l_1 + 1}(a) \rightarrow \cdots \sigma^{l_2 - 1}(a) &\rightarrow n+2, \\
\cdots &\rightarrow \cdots \\
\sigma^{l_{s-1}}(a) \rightarrow \sigma^{l_{s-1} + 1}(a) \rightarrow \cdots \sigma^{l_s - 1}(a) &\rightarrow n+s,
\end{aligned}
$$

where $l_1 + l_2 + \cdots l_s = \mu_i$ and by convention the corresponding path is empty if $l_i = 0$. Obviously there are $\mu_i \binom{\mu_i + s - 1}{s-1}$ ways to form such s paths starting from any cycle of length μ_i. Finally we complete the construction of f: $[n] \rightarrow [n+s]$ by using the s paths above and the remaining $l(\mu) - 1$ cycles of σ. This gives the left-hand side of (1).

The theorem follows then from the two above countings.

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